CS 126 Lecture P6: Recursion

Why Learn Recursion?

- Master a powerful programming tool
- Gain insight of how programs (function calls) work

Outline

- <u>**What is recursion?**</u>
- How does it work?
- Examples

COS 126 Lecture P6: Recursion

Recursive program: one that calls itself

MATHEMATICAL INDUCTION:

- To prove $S(N)$
  - * prove $S(0)$
  - * prove $S(N)$, assuming $S(k)$ for all $k<N$

Ex: "triangle numbers"

$0 + 1 + 2 + 3 + \ldots + N = N(N+1)/2$

- * trivially true for $N = 0$
- * $0 + 1 + 2 + 3 + \ldots + N$

$$= 0 + 1 + 2 + \ldots + N-1 + N$$
$$= (N-1)N/2 + N \;=\; N(N+1)/2$$

RECURSION:

- To compute $f(N)$
  - * compute $f(0)$
  - * compute $f(N)$, using $f(k)$ for $k<N$

Ex: triangle numbers

```
int tri(int N)
  {
    if (N == 0) return 0;
    return N + tri(N-1);
  }
```

Number conversion

- To convert an integer N to binary:
  - stop if N is 0
  - write "1" if N odd, "0" if N even
  - move left one position
  - convert N/2

Ex:

```
.       42        0
.       21       10
.       10      010
.        5     1010
.        2    01010
.        1   101010
```

check:

```
.          5    4    3    2    1    0
.          1    0    1    0    1    0
.
```

$2^5 + 2^3 + 2^1$

$32 + 8 + 2 = 42$

- Easiest way to convert to binary by hand
- Corresponds directly to a recursive program

Recursive number conversion

- Computer prints from left to right
  - need to convert N/2, then print right bit

```
void convert(int N)
   {
      if (N/2 > 0) convert(N/2);
      printf("%c", '0'+ N % 2);
   }
```

Proof of correctness: $N = 2*(N / 2) + (N \% 2)$

```
convert(42)
   convert(21)
      convert(10)
         convert(5)
            convert(2)
               convert(1)
                  printf("1")
               printf("0")
            printf("1")
         printf("0")
      printf("1")
   printf("0")
```

Indentation level denotes statements belonging to same "invocation"

1
0
1
0
1
0

→ 101010 = 42

- Works to convert to any base
  (change "2" to "b" everywhere in code)

Demo `convert()`

Outline

- ~~What is recursion?~~
- **How does it work?**
- Examples

Function "Environment"

- When a function executes, it lives in an "environment"
- What's an "environment"?
  - Value of local variables (scratch space)
  - Which statement the computer is executing currently

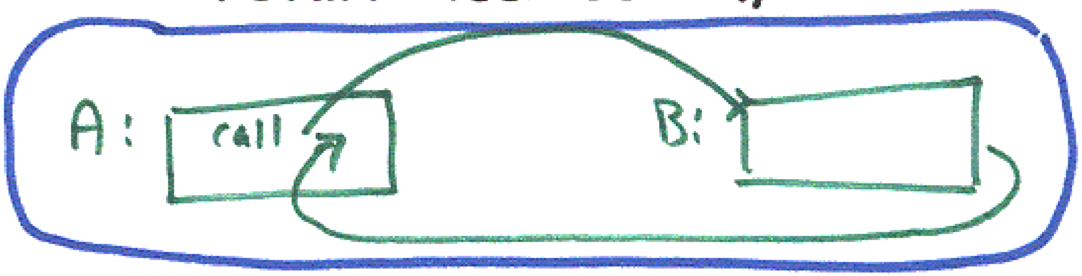
Implementing Recursion

Any function call requires system to

- set the values of the parameters
- save the "environment"
- jump to the first instruction in the function

  execute the function
- restore the "environment"
- continue at the instruction after the call

  "return address" (part of environment)

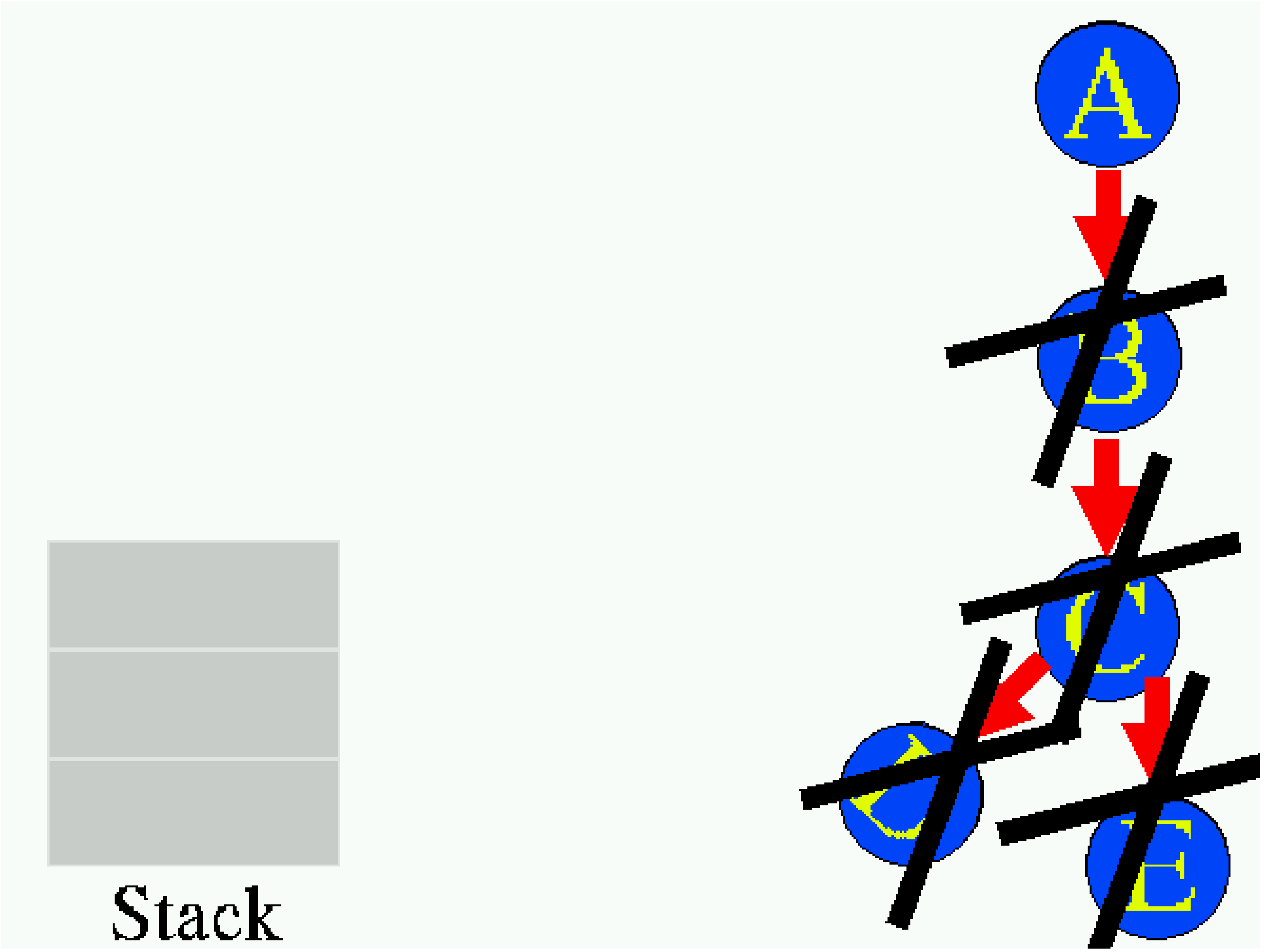


- Use pushdown stack for save/restore

  call: push environment

  return: restore environment from stack

Demo Use of Stacks to Implement Function Calls



Stack details for number conversion

- "Environment" is value of N
  - function call: push N to stack
  - return: pop stack to N

	N	stack contents
.		
.convert(42)		42
. convert(21)		42 21
. convert(10)		42 21 10
. convert(5)		42 21 10 5
. convert(2)		42 21 10 5 2
. convert(1)		
. printf("1")	1	
. [return for 1]	2	42 21 10 5
. printf("0")		
. [return for 2]	5	42 21 10
. printf("1")		
. [return for 5]	10	42 21
. printf("0")		
. [return for 10]	21	42
. printf("1")		
. [return for 21]	42	
. printf("0")		
. [return for 42]		

popped from stack

Removing Recursion

```
{
    base case;
    some code;
    recursion;
    more code;
}
```

```
{
    repeat {
        some code;
        push environment;
    }
    base case;
    repeat {
        pop environment;
        more code;
    }
}
```

- We can remove recursion from any function by using an explicit stack
- Helps us understand nature of the computation (no other reason to do so)

Removing Recursion

```
{
    base case;
    some code;
    recursion;
    more code;
}
```

→

```
{
  repeat {
    some code;
    push environment;
  }
  base case;
  repeat {
    pop environment;
    more code;
  }
}
```

Ex: number conversion

```
void convert(int N)
  {
     STACKinit();
     while (N > 0) { STACKpush(N); N = N/2; }
     while (!STACKempty())
        printf("%c", '0'+ STACKpop() % 2);
  }
```

Tail Recursion

```
int tri(int N)
   {
      if (N == 0) return 0;
      return N + tri(N-1);
   }
```

```
int tri(int N)
   { int t;
      for (t = 0; N > 0; N++) t += N;
      return t;
   }
```

- If single recursive call is the last action, don't need a stack
- Why?
  - nothing to do after recursion => no need to remember stuff => no need for stack

Possible Pitfall with Recursion

Simple recursive programs
can consume excessive resources

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5
15

Ex: Compute binomial coefficients

```
int f(int N, int k)
   {
      if ((k < 0) || (k > N)) return 0;
      if (N == 0) return 1;
      return f(N-1, k) + f(N-1, k-1);
   }
```

- Seems to run for a long time to compute f(30, 15).

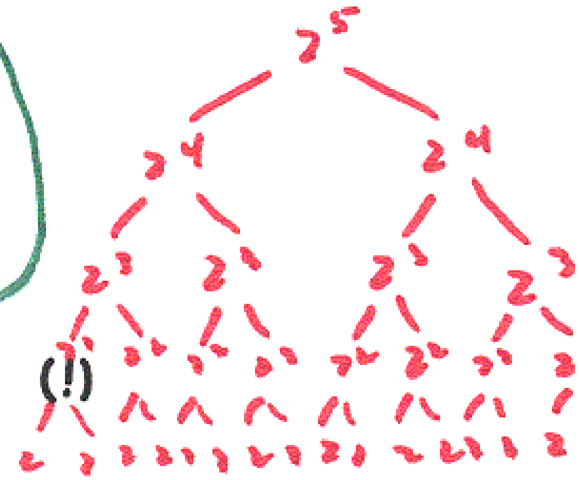
Q: Why?
A: Recomputes intermediate results

Possible Pitfall with Recursion

- Simpler example: hard way to compute 2^N

```
int f(int N)
  {
    if (N == 0) return 1;
    return f(N-1)+f(N-1);
  }
```

Takes time proportional to 2^N (!)



- DO NOT use these programs!

Solution: DYNAMIC PROGRAMMING

save away intermediate results

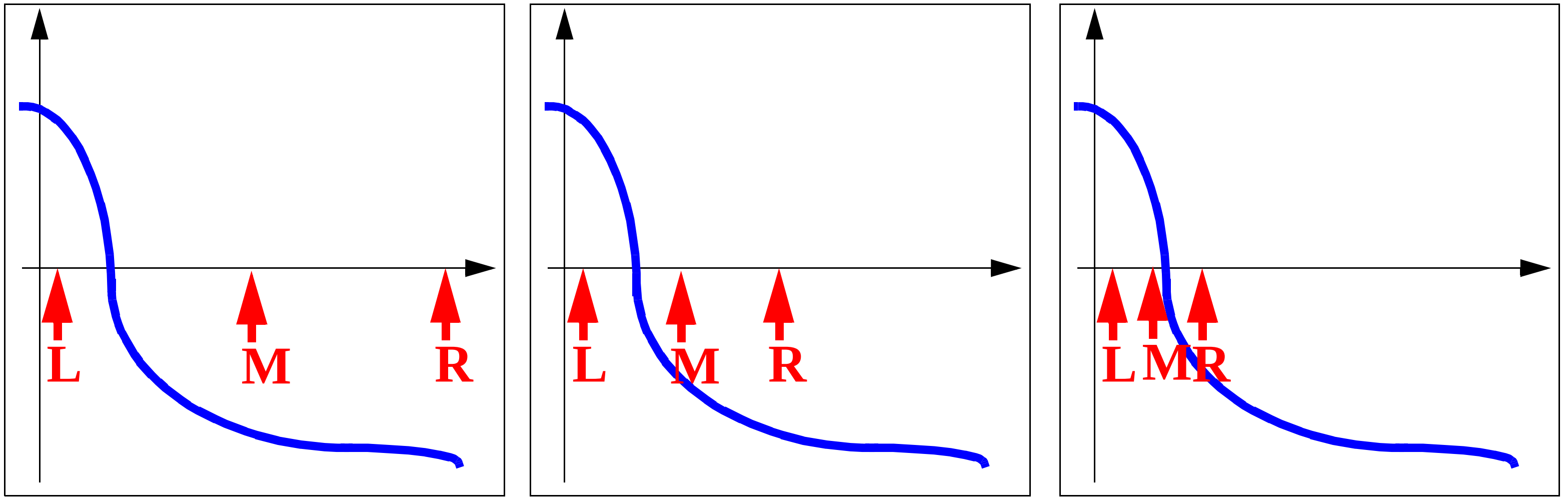
see Sedgewick, section 5.2

P6.9

Outline

- ~~What is recursion?~~
- ~~How does it work?~~
- **Examples**

Divide-and-Conquer



Many computations are
naturally expressed as recursive programs

ITERATION
another way to write "for" loop

"DIVIDE and CONQUER"
solve a problem by dividing into smaller ones

Ex: root finding via "bisection"

Finding Root via Bisection

```
float bisectr(float l, float r)
  {
    float m;
    m = (l + r) / 2;
    if ((r - l) < epsilon) return m;
    if (f(m) > 0.0)
      return bisectr(m, r);
    else
      return bisectr(l, m);
  }
```

Bisection for Integer Functions

```
int bisectr(int l, int r)
  {
     int m;
     m = (l + r) / 2;
     if (f(m) == 0) return m;
     if (r <= l) return -1;
     if (f(m) > 0)
        return bisectr(m+1, r);
     else
        return bisectr(l, m-1);
  }
```

Binary Search

Suppose an array A has N integers, in order

x	0	1	2	3	4	5	6	7	8	9	10	11	12
f(x)	1	1	2	5	8	13	21	34	55	89	144	233	377

SEARCH PROBLEM: is a given integer v in A?

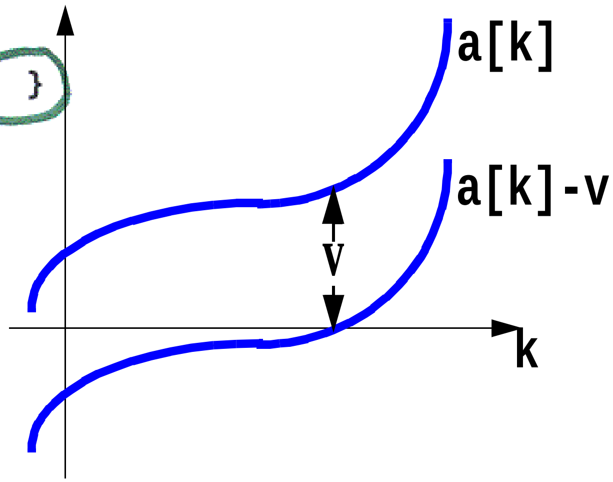
- Observations:
  - An array is a function mapping integer indices to contents
  - A sorted array is a monotonically increasing function

SOLUTION: use above program with

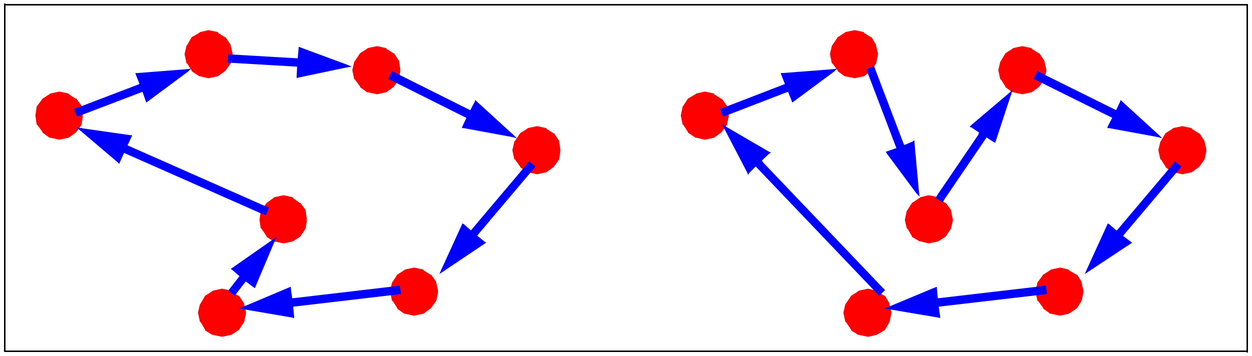
```
int f(int k) { return v - a[k]; }
```

Ex: v = 144

.	l	r	m	f(m)
.	0	12	6	+
.	7	12	9	+
.	10	12	11	-
.	10	10	10	0



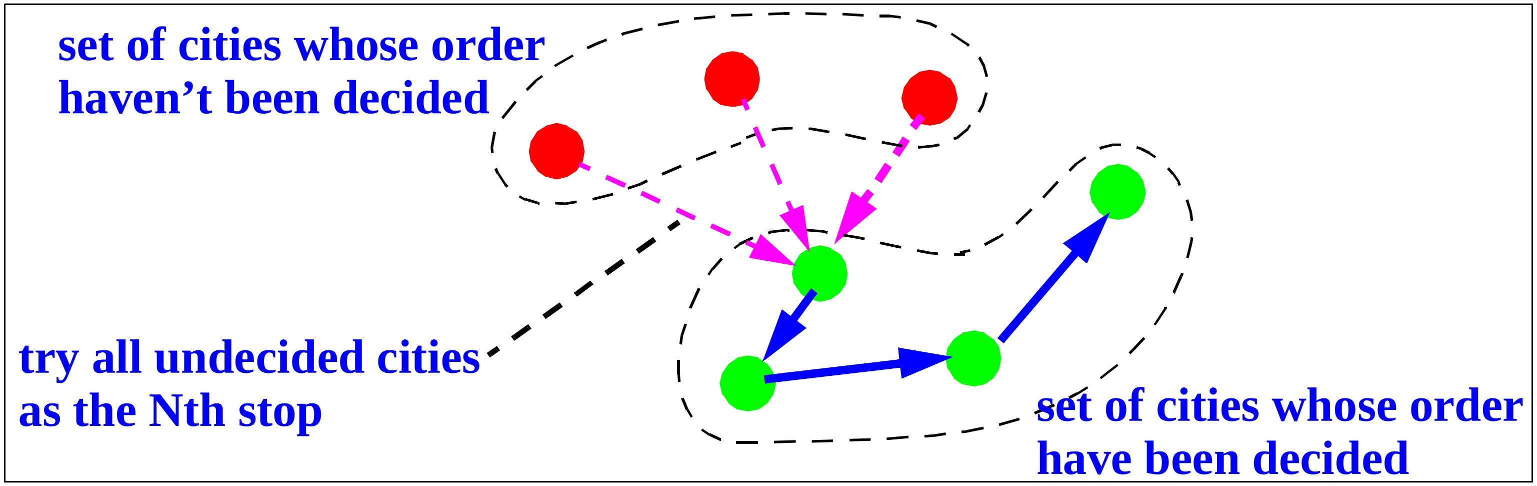
Traveling Salesman Problem



Given a set of points, find the shortest tour connecting all the points

- Recursive solution for trying all possibilities

Traveling Salesman Problem



- Recursive solution for trying all possibilities

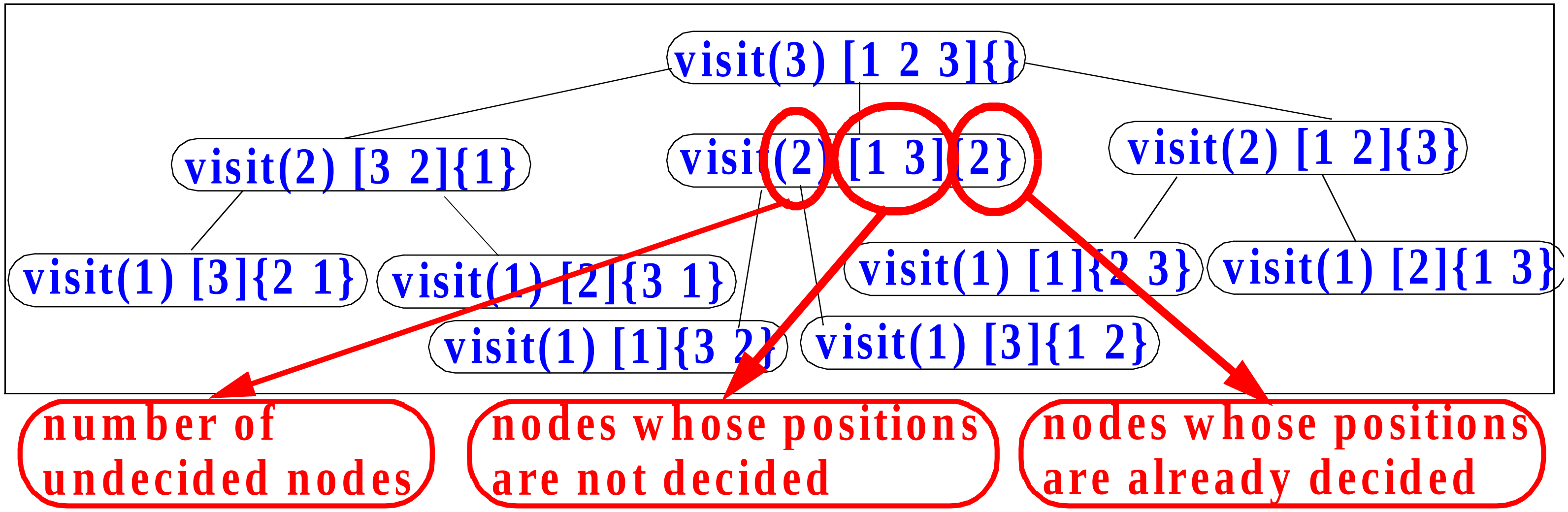
```
visit(int N)
  {
      int i;
      if ( N == 1) { checklength(); return; }
      for (i = 1; i <= N; i++)
        {
          swap(i, N);
          visit(N-1);
          swap(i, N);
        }
  }
```

Number of nodes whose positions have not been decided

Visit ith city as the last (Nth) step

Decide the positions of the other undecided cities

Traveling Salesman Problem



- Takes N! steps

- Can't run for very large N

  no computer can ever run this

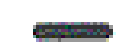
  to completion for N = 100 [100! > 10^150]

[stay tuned]

Recursion: Dragon Curve

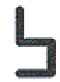
Fold a strip of paper in half n times
unfold to right angles

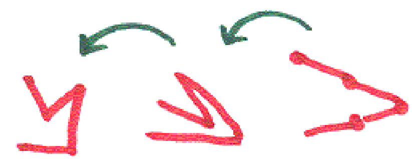
$n = 0$



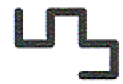
$n = 1$

$n = 2$

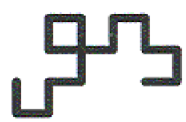




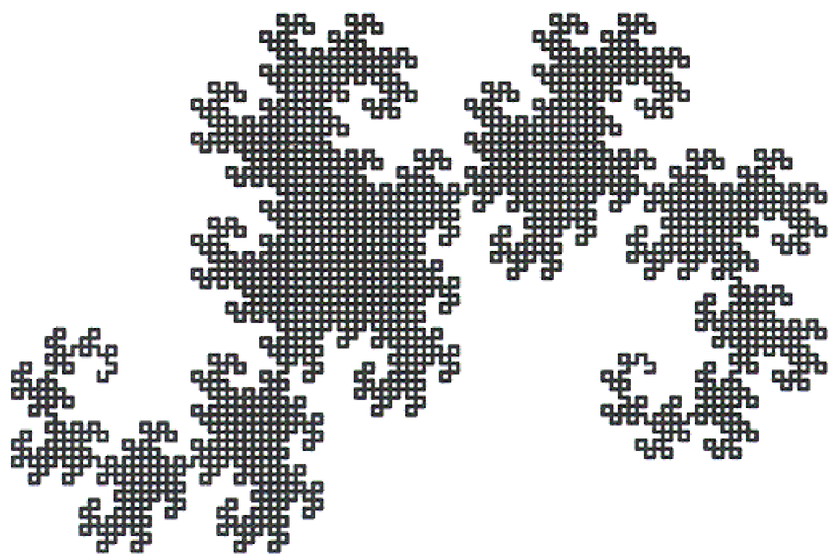
$n = 3$



$n = 4$



$n = 12$



Drawing a Dragon Curve

Use simplest turtle graphics

F: move forward one step (pen down)

L: turn left

R: turn right

n = 0

F

n = 1

F L F

n = 2

F L F L F R F

L L R

n = 3

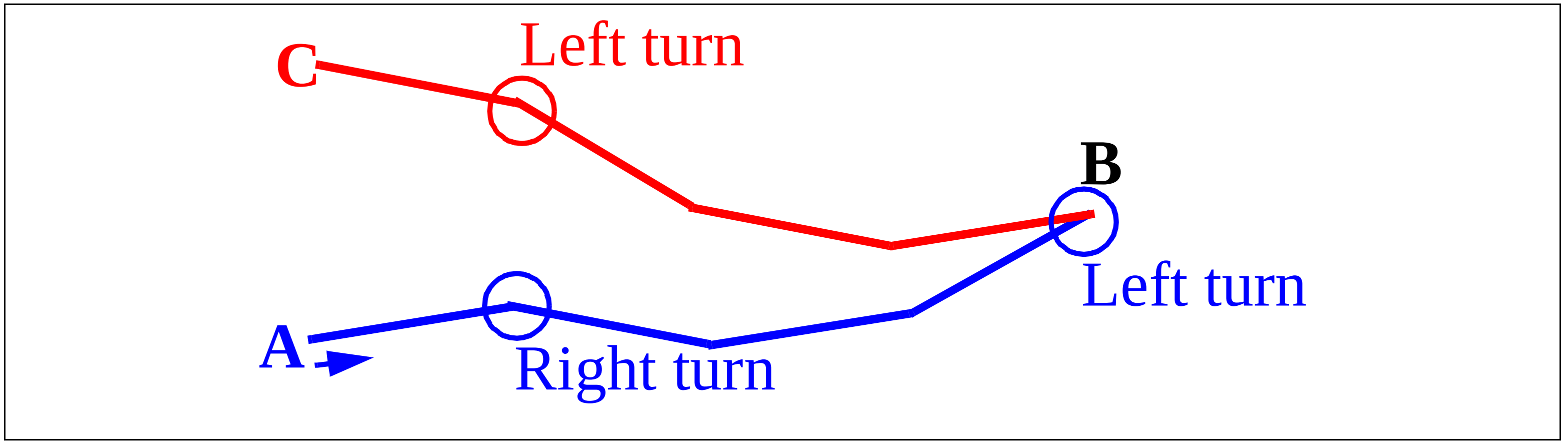
F L F L F R F L F L F R F R F

L L R L L R R

n = 4

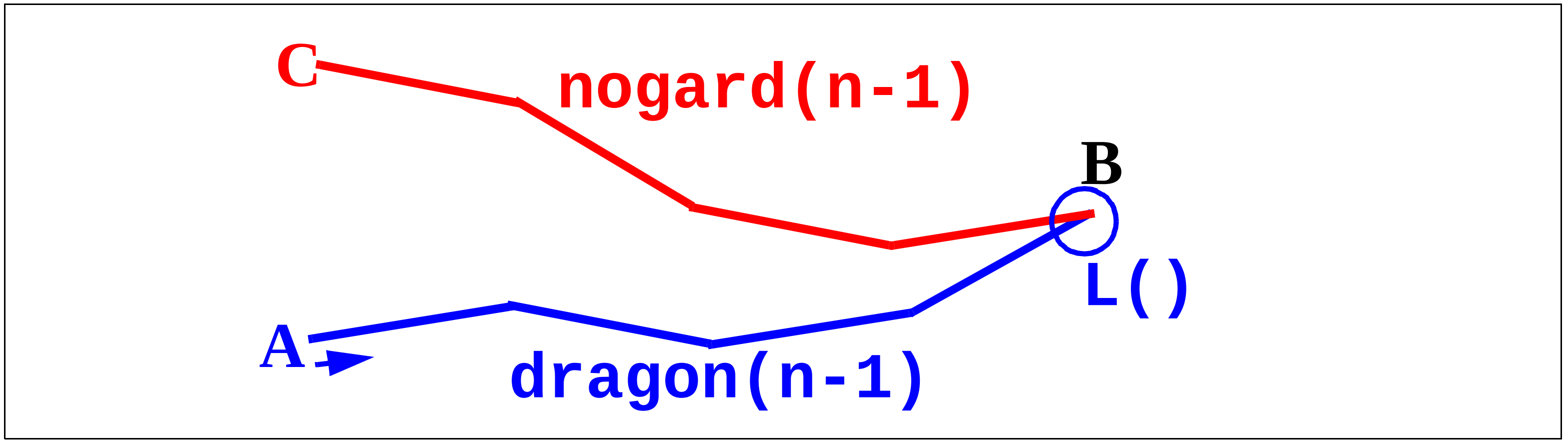
L L R L L R R L L L R R L R R

Intuition of Algorithm



- $\overline{AB}$ is a smaller dragon curve by itself
- $\overline{CB} = \overline{AB}$
- Therefore $\overline{BC}$ is the reverse of $\overline{AB}$
- Therefore every turn along $\overline{BC}$ is the opposite of the corresponding turn on $\overline{AB}$

Recursive Program for Dragon Curve




```
dragon(int n)
{
   if (n == 0) { F(); return; }
   dragon(n-1);
   L();
   nogard(n-1);
}
```

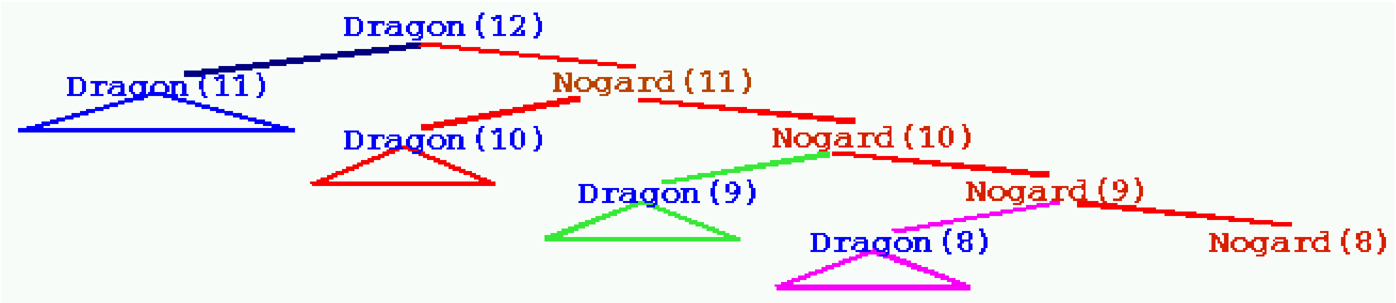
Backwards Dragon Curve

```
dragon(int n)
{
  if (n == 0) { F(); return; }
  dragon(n-1);
  L();
  nogard(n-1);
}
```

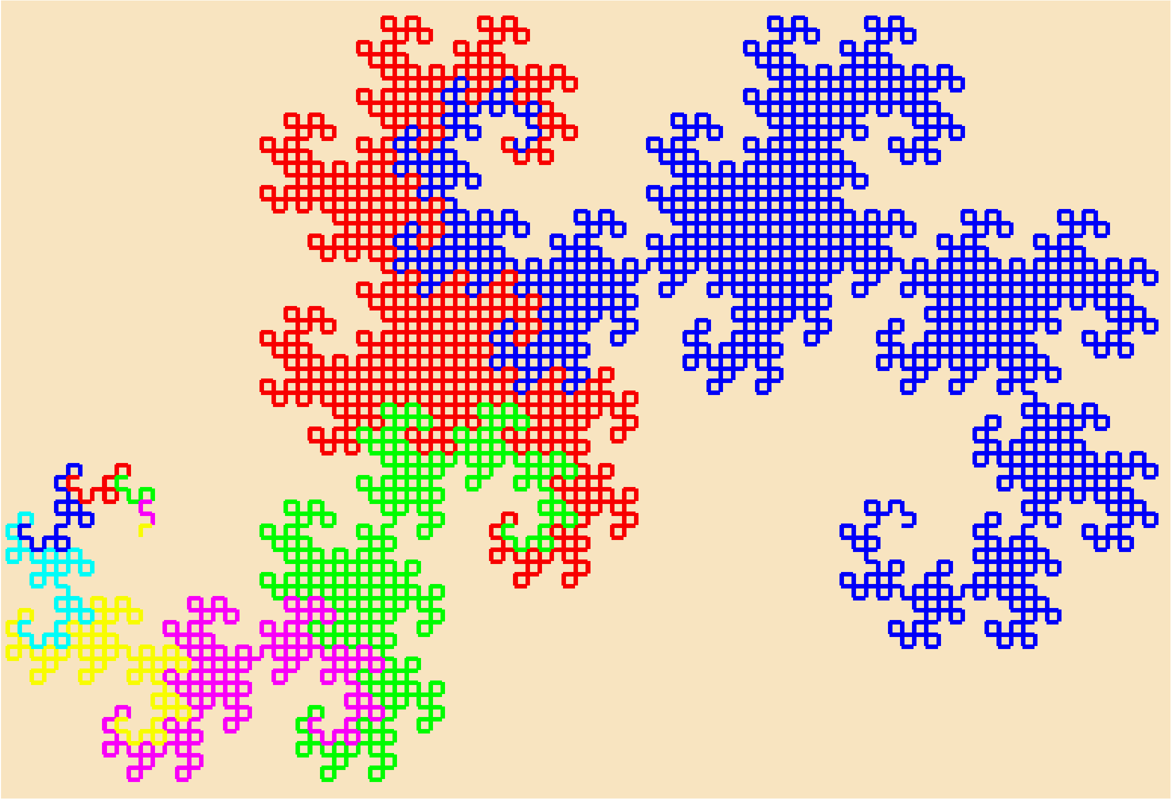
Reverse

```
nogard(int n)
{
  if (n == 0) { F(); return; }
  dragon(n-1);
  R();
  nogard(n-1);
}
```

dragon Demo



Dragon(12)
Dragon(11)
Nogard(11)
Dragon(10)
Nogard(10)
Dragon(9)
Nogard(9)
Dragon(8)
Nogard(8)



Alternate "dragon"

- Replace call to "nogard"
  by nonrecursive version

```
dragon(int n)
{
   int k;
   if (n == 0)  { F(); return; }
   dragon(n-1);
   L();
   for (k = n-2; k >= 0; k--)
     {
       dragon(k);
       R();
     }
   F();
}
```

$$\overbrace{F\ L\ F\ L\ F\ R\ F\ L\ F\ L\ F\ R\ F\ R\ F}^{D(3)}$$

$$\underbrace{F\ L\ F\ L\ F\ R\ F}_{D(2)}\ L\ \underbrace{F\ L\ F}_{D(1)}\ R\ \underbrace{F}_{D(0)}\ R\ F$$

- Points out self-similarities in curve

PostScript dragon curve

- Easy to implement because of built-in
  - turtle graphics
  - stack (dup repliates stack top)
- Passing args to recursive functions is tricky

all arguments and "scratch variables" are on the stack!

```
/L { 90 rotate } def
/R { 90 neg rotate } def
/F { 2 0 rlineto } def
/dragon
   { dup 0 eq
        { F pop }
        { 1 sub dup dragon L nogard }
     ifelse
   } def
/nogard
   { dup 0 eq
        { F pop }
        { 1 sub dup dragon R nogard }
     ifelse
   } def
200 400 moveto
15 dragon
stroke
showpage
```

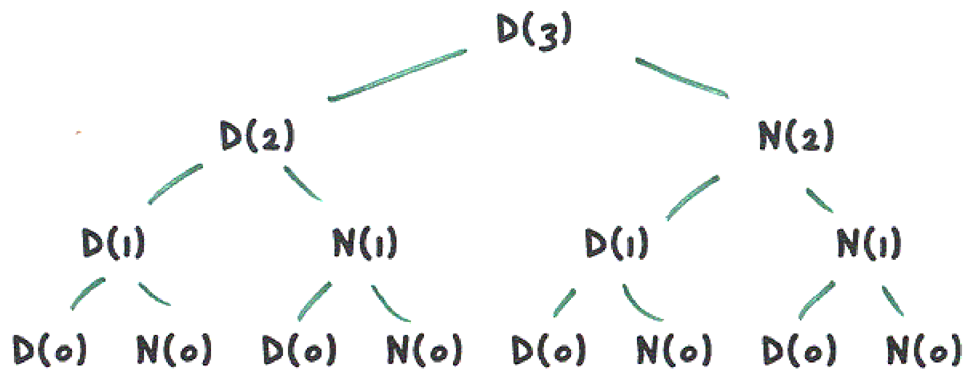
replicates top before popping for comparison

pushes two copies of (n-1) for the two recursive calls

CAUTION:

2^N line segments in curve of order N

Nonrecursive dragon curve



F L F L F R F L F L F R F R F

To write down the whole dragon curve sequence
- * first, put "F" in every other space
- * put "L", "R" (alternating) in every other remaining space
- * continue until done

```
F   F   F   F   F   F   F   F
F L F   F R F   F L F   F R F
F L F L F R F   F L F R F R F
F L F L F R F L F L F R F R F
```

- Like Towers of Hanoi (see Sedgewick, section 5.2)
  requires too much storage (how much?)
  "ruler function" connects to binary numbers
  Details? [challenge for the bored]

step i : 1 : if the bit to the left of the

What We Have Learned

- How recursion works
  - A recursive call is no different from a “regular” call
  - It involves saving the old environment for later return
- Learn to trace the execution of given recursive programs (using pictures)
- Learn to write simple recursion
  - What’s the base case?
  - What’s the induction case?